

Math 107: Calculus II, Spring 2015: Midterm Exam II
Monday, April 13 2015

Give your **name**, TA and section number:

Name:

TA:

Section number:

1. There are 5 questions for a total of 100 points. The value of each part of each question is stated.
2. **Do not open your booklet until told to begin.** The exam will be 50 minutes long.
3. You may **not** use phones, calculators, books, notes or any other paper. Write all your answers on this booklet. If you need more space, you can use the back of the pages.
4. **Unless specified otherwise, you must show ALL your working and explain your answers clearly to obtain full credit!**
5. **Read the questions carefully!** Make sure you understand what each question asks of you.

Please read the following statement and then sign and date it:

“I agree to complete this exam without unauthorized assistance from any person, materials, or device.”

Signature:

Date:

Question	Points	Score
1	30	
2	20	
3	12	
4	18	
5	20	
Total:	100	

1. Let X be a discrete random variable with range $\{0, 1, 2, 3, 4\}$ and probability mass function

$$P(X = 0) = \frac{1}{9}, \quad P(X = 1) = \frac{2}{9}, \quad P(X = 2) = \frac{3}{9}, \quad P(X = 3) = \frac{2}{9}, \quad P(X = 4) = \frac{1}{9}.$$

For the following questions, carry out the computations and leave your answer as an irreducible fraction.

- (a) (12 points) Find $E(X)$ and $\text{var}(X)$. **Show all your work!**

$$\begin{aligned} E(X) &= \sum_{k=0}^4 k \cdot P(X = k) \\ &= 0 \cdot \frac{1}{9} + 1 \cdot \frac{2}{9} + 2 \cdot \frac{3}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{1}{9} \\ &= 2 \\ \text{var}(X) &= \sum_{k=0}^4 (k - 2)^2 \cdot P(X = k) \\ &= (-2)^2 \cdot \frac{1}{9} + (-1)^2 \cdot \frac{2}{9} + 0^2 \cdot \frac{3}{9} + 1^2 \cdot \frac{2}{9} + 2^2 \cdot \frac{1}{9} \\ &= \frac{12}{9} = \frac{4}{3} \end{aligned}$$

- (b) (8 points) Find $P(1 \leq X \leq 3)$. **Show all your work!**

$$\begin{aligned} P(1 \leq X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \frac{2}{9} + \frac{3}{9} + \frac{2}{9} = \frac{7}{9} \end{aligned}$$

Alternatively, you could do

$$\begin{aligned} P(1 \leq X \leq 3) &= 1 - P(X = 0) - P(X = 4) \\ &= 1 - \frac{1}{9} - \frac{1}{9} = \frac{7}{9} \end{aligned}$$

- (c) (10 points) We measure X four times independently. What is the probability that $X = 2$ for exactly two of the four measurements? **Show all your work!**

We have 4 trials (the 4 measurements), and we can think of the outcome $X = 2$ as success and $X \neq 2$ as failure. There are $\binom{4}{2}$ ways of getting 2 success in 4 trials, the probability of each success is $P(X = 2) = \frac{3}{9} = \frac{1}{3}$, and the probability of failure is $P(X \neq 2) = 1 - \frac{1}{3} = \frac{2}{3}$. Thus the probability of getting $X = 2$ exactly 2 out of 4 times is:

$$\binom{4}{2} \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = 6 \cdot \frac{4}{9 \cdot 9} = \frac{8}{27}.$$

NOTE: This question was the same as the one in the practice slides, just with different numbers.

2. Suppose that a continuous random variable X has distribution function

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Find $P(X \leq 5)$. **Show all your work!**

$$\begin{aligned} P(X \leq 5) &= \int_{-\infty}^5 f(x) dx \\ &= \int_{-\infty}^0 0 dx + \int_0^5 e^{-x} dx \\ &= (-e^{-x}) \Big|_0^5 = 1 - \frac{1}{e^5}. \end{aligned}$$

(b) (15 points) Find $E(X)$. **Show all your work!**

$$E(X) = \int_{-\infty}^{\infty} x e^{-x} dx = \int_0^{\infty} x e^{-x} dx$$

since $f(x) = 0$ for $x \leq 0$. Thus,

$$\begin{aligned} E(X) &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx = \lim_{b \rightarrow \infty} (-x e^{-x} - e^{-x}) \Big|_0^b \\ &= \lim_{b \rightarrow \infty} (-b e^{-b} - e^{-b} + 1) = 1 - \lim_{b \rightarrow \infty} b e^{-b} - \lim_{b \rightarrow \infty} e^{-b}. \end{aligned}$$

By L'Hopital,

$$\lim_{b \rightarrow \infty} b e^{-b} = \lim_{b \rightarrow \infty} \frac{b}{e^b} \stackrel{\infty/\infty}{=} \lim_{b \rightarrow \infty} \frac{1}{e^b} = 0.$$

Also,

$$\lim_{b \rightarrow \infty} e^{-b} = 0, \text{ so } E(X) = 1.$$

NOTE: This question was also in the practice slides, with e^{-2x} instead of e^{-x} , and it was part of a long homework question you had to solve where you had e^{-ax} for a parameter a .

3. (12 points) The following pictures, in some order, denote the image of the vector $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ under the transformations given by the matrices.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Match the pictures with the corresponding matrix by writing A_i for the correct value of i in the line next to the picture. **You do not have to show any work.**

I am sorry, I don't have the pictures.

(a) A_2 , because

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

(b) A_4 , because

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

(c) A_3 , because

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

(d) A_1 , because

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

4. (18 points) Find the general solution to the differential equation

$$\frac{dy}{dx} = y(1 - y).$$

Show all your work! You will be graded on the completeness and quality of your solution.

First note that the constant functions $y(x) = 0$ and $y(x) = 1$ satisfy the differential equation, therefore they are solutions.

Now, assume that $y(x) \neq 0, 1$, then we can divide by $y(1 - y)$ to get

$$\frac{1}{y(x)(1 - y(x))} y'(x) = 1.$$

Integrating we get,

$$\int \frac{1}{y(x)(1 - y(x))} y'(x) dx = \int 1 dx.$$

Now we can use the substitution of variables

$$y = y(x), \quad dy = y'(x) dx,$$

to get

$$\int \frac{1}{y(y - 1)} = \int dx.$$

NOTE: You did not lose points if you didn't write out the substitution explicitly and separated variables to get

$$\int \frac{1}{y(y - 1)} = \int dx.$$

However, if you wrote

$$\frac{1}{y(y - 1)} = dx.$$

that is a *crucial* mistake that shows profound misunderstanding. Alex was very generous in grading that, and didn't subtract nearly as many points as I would have. If you have written that, please think about it carefully. $\frac{dx}{dx}$ means $y'(x)$, the derivative of y , which is a function of x . You cannot split up the dx and multiply by it - that's not a number and that makes no sense. I know some high school books say that, but I told you that's dumming down the procedure for obtaining the answer and you should not buy that. Please think carefully about how we derived the separation of variables we are using. I was sad to see some students still write that.

We use partial fraction decomposition to compute

$$\int \frac{dy}{y(1 - y)} = \int \frac{dy}{y} + \int \frac{dy}{1 - y} = \ln |y| - \ln |1 - y| + C,$$

for C an arbitrary real constant Thus

$$\ln |y| - \ln |1 - y| = x + C$$

for C an arbitrary real constant. So,

$$\ln \left| \frac{y}{1 - y} \right| = x + C$$

for C an arbitrary real constant. So,

$$\left| \frac{y}{1-y} \right| = e^C e^x$$

for C an arbitrary real constant (so e^C can range through all positive real numbers). So,

$$\frac{y}{1-y} = \pm e^C e^x,$$

i.e.,

$$\frac{y}{1-y} = C e^x,$$

for C an arbitrary non-zero constant (since $\pm e^C$ ranges through all nonzero real numbers as C ranges through all the real numbers.)

Now we have to solve for y :

$$y = (1-y)C e^x \Rightarrow y(1 + C e^x) = C e^x,$$

so the general solution of our differential equation is

$$y(x) = \frac{C e^x}{1 + C e^x}, \text{ for any } C \neq 0, \text{ and the constant functions } y(x) = 0 \text{ and } y(x) = 1.$$

NOTE: This is just a particular case of the logistics equation, with all constants being equal to 1.

NOTE: It was important to not forget the trivial solutions. I emphasized that in class even when we had an initial value condition - we always checked the constant solutions first that we disregard when we separate variables (if we had an initial constraint that they didn't satisfy, we dismissed them.) Also, that's why I put the question on the review slides where the solution turned out to be precisely the constant solution.

5. Let $A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$.

- (a) (5 points) State the definition of what it means for a scalar λ to be an eigenvalue of A .

A scalar λ is an eigenvalue of A if there exists a *nonzero* vector v such that $Av = \lambda v$.

NOTE: That's it. That's all you had to write.

“NONZERO” is the essential part of this definition. If you don't say that, then $Av = \lambda v$ is trivially satisfied by the zero vector for any real number λ , so any real number would be an eigenvalue. I stressed this over and over again in class.

- (b) (15 points) Find the eigenvalues of A (start from the definition, carry out and explain all the steps in your argument.)

Show all your work! You will be graded on the completeness and quality of your solution.

A scalar λ is an eigenvalue of A if there exists a *nonzero* vector $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $Av = \lambda v$.

Now

$$\begin{aligned} Av = \lambda v &\iff Av - \lambda v = 0 \\ &\iff Av - \lambda Iv = 0 \\ &\iff (A - \lambda I)v = 0 \\ &\iff \left[\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff \begin{pmatrix} 3 - \lambda & 0 \\ 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Thus λ is an eigenvalue if and only if the last equation has a nonzero solution, i.e., if there is $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which satisfies it.

This equation has a nonzero solution if and only if the matrix $\begin{pmatrix} 3 - \lambda & 0 \\ 2 & 1 - \lambda \end{pmatrix}$ is NOT invertible – if the inverse existed, then we could multiply by it on both sides to get $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and that would be the unique solution.

The matrix $\begin{pmatrix} 3 - \lambda & 0 \\ 2 & 1 - \lambda \end{pmatrix}$ is not invertible if and only if $\det \begin{pmatrix} 3 - \lambda & 0 \\ 2 & 1 - \lambda \end{pmatrix} = 0$. Solving this, we get

$$(3 - \lambda)(1 - \lambda) - 2 \cdot 0 = 0,$$

so

$$\lambda = 3 \text{ or } 1.$$

NOTE: The essential part was to realize that we need a NONZERO solution to

$$\begin{pmatrix} 3 - \lambda & 0 \\ 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and to know that that only happens if the coefficient matrix is not invertible, *and that is why* we need its determinant to be 0. Unless you explicitly said these things, your solution is not complete.

NOTE: When you distribute and pull out v from $Av - \lambda Iv = 0$, you need to keep it on the right: $(A - \lambda I)v = 0$. Note that

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

is not even defined !!

And also, it's not true that $(A - B)C = CA - CB$, even if the multiplications are defined! What is true is that $(A - B)C = AC - BC$, but we don't have that $AC = CA$ or $BC = CB$.